

NOTE ON SHEAF THEORY

YASSINE AIT MOHAMED

ABSTRACT

In this note, we review some fundamental concepts and constructions in sheaf theory.

CONTENTS

1. Presheaves	1
1.1. Morphisms of presheaves	3
2. Sheaves	4
2.1. Morphisms of sheaves	5
3. Sheafification	6
4. Subsheaves and Quotient sheaves	8
4.1. Subsheaves	8
4.2. Quotients sheaves	9
5. Continuous maps and sheaves	10
5.1. Pushforward	10
5.2. Pullback	11
6. Exact sequences of sheaves	12
7. Glueing sheaves	14

1. Presheaves

Notation. Let X be a topological space. We denote by $\mathcal{T}_X$ the category having for objects the open subsets of X and for morphisms identity maps and inclusions.

$\mathcal{C}$ will denote a category, which can be the category of sets (also denoted by $\mathcal{Set}$), that of groups (also denoted by $\mathcal{Gp}$), that of R -modules (also denoted by $\mathcal{Ring}$), that of R -modules (also denoted by $R\text{-}\mathcal{Alg}$), for some ring R .

Definition 1.1. Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of the following data :

- i) For every open subset U of X , a set $\mathcal{F}(U)$.
- ii) Whenever $U \subseteq V$ are two open subsets of X , a map

$$\text{res}_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$

called the restriction map, which satisfies the following conditions :

- a) $\text{res}_{U,U} = \text{id}_{\mathcal{F}(U)}$.
- b) Having three open subsets $U \subseteq V \subseteq W$ of X , then $\text{res}_{V,U} \circ \text{res}_{W,V} = \text{res}_{W,U}$

Remarks 1.2. 1) We will mostly write $s|_U$ for s when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called sections of (the presheaf $\mathcal{F}$) over U .

2) By considering $\mathcal{F}(U)$ as object in some category $\mathcal{C}$ and assuming that $\text{res}_{V,U}$ is a morphism between the objects $\mathcal{F}(V)$ and $\mathcal{F}(U)$, we may define more generally a presheaf $\mathcal{F}$ on X into $\mathcal{C}$.

3) Note that we can state definition 1.1 in another way: Let X be a topological space. A presheaf $\mathcal{F}$ on X (into a category $\mathcal{C}$) is a contravariant functor from $\mathcal{T}_X$ into $\mathcal{C}$.

$$\begin{array}{ccc} \mathcal{F} : & \mathcal{T}_X & \longrightarrow & \mathcal{C} \\ & U & \longmapsto & \mathcal{F}(U) \end{array}$$

Examples 1.3. 1) For a topological space, a presheaf $\mathcal{C}_X$ of $\mathbb{R}$ -algebras on X is defined by assigning to every open $U \subseteq X$ the set of continuous functions $U \longrightarrow \mathbb{R}$.

2) Let X be a variety, we previously considered the presheaf of k -algebras $\mathcal{O}_X$. For any open $U \subseteq X$, $\mathcal{O}_X(U)$ is the k -algebra of regular functions. If X be an affine variety we have $\mathcal{O}_X(U) = k[U]$.

3) Let X be a topological space, the formula :

$$U \longmapsto \begin{cases} \mathbb{Z} & \text{if } U = X \\ \{0\} & \text{otherwise} \end{cases}$$

defines a presheaf of abelian groups on X .

Although it is possible to define a presheaf of a topological space X into an arbitrary category $\mathcal{C}$, we will be interested in what follows only in cases where the objects of $\mathcal{C}$ are sets (that could have an additional structure) and the morphisms $\text{res}_{V,U}$ are maps (which are morphisms for the extra structure on $\mathcal{F}(V)$ and $\mathcal{F}(U)$).

Definition 1.4. Let $\mathcal{F}$ be a presheaf on X , a subpresheaf $\mathcal{G}$ (of $\mathcal{F}$) is a presheaf on X such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for every open $U \subseteq X$, and such that the restriction maps of $\mathcal{G}$ are induced by those of $\mathcal{F}$.

Example 1.5. If U is an open subset of X , every presheaf $\mathcal{F}$ on X induces, in an obvious way, a presheaf $\mathcal{F}_U$ on U by setting $\mathcal{F}_U(V) = \mathcal{F}(V)$ for every open subset V of U . This is the restriction of $\mathcal{F}$ to U .

1.1. Morphisms of presheaves.

Definition 1.6. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X . A morphism of presheaves ψ from $\mathcal{F}$ to $\mathcal{G}$ consists of the datum, for all open U of X , of a morphism $\psi(U)$ from $\mathcal{F}(U)$ to $\mathcal{G}(U)$, so that the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\psi(V)} & \mathcal{G}(V) \\ \text{res}_{V,U} \downarrow & & \downarrow \text{res}_{V,U} \\ \mathcal{F}(U) & \xrightarrow{\psi(U)} & \mathcal{G}(U) \end{array}$$

commutes for any pair (U, V) of open subsets of X such that $U \subseteq V$.

Remarks 1.7. 1) The commutativity of the diagram is written : $\psi(V)(s)|_U = \psi(U)(S|_U)$, where $s \in \mathcal{F}(V)$.

- 2) Morphisms of presheaves can be composed. So that presheaves on the topological space X form a category, that we will denote by PreSh_X .
- 3) A morphism $\psi : \mathcal{F} \rightarrow \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ is an isomorphism if it has a two-sided inverse i.e, a morphism $\phi : \mathcal{G} \rightarrow \mathcal{F}$ such that $\psi \circ \phi = \text{id}_{\mathcal{G}}$ and $\phi \circ \psi = \text{id}_{\mathcal{F}}$.

Definition 1.8. Assume $\mathcal{C}$ has direct limits. The stalk of a presheaf $\mathcal{F}$ at a point $x \in X$ is

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

The direct limit is taken over open neighborhoods of x , and restriction maps between them. Given a section $s \in \mathcal{F}(U)$, and a point $x \in U$, we let $s_x \in \mathcal{F}_x$ denote the image of s under the natural morphism

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x \\ s & \longmapsto & s_x \end{array}$$

An element of the stalk is called a germ.

More generally, if $Y \subseteq X$ is a closed and irreducible subset. Then, we set

$$\mathcal{F}_Y := \varinjlim_{U \cap Y \neq \emptyset} \mathcal{F}(U)$$

Notation. Let X be a topological space and $x \in X$, we denote by $\mathcal{V}$ the set of open neighborhoods of x , which is filtering for the opposite order to inclusion i.e, for all $U, V \in \mathcal{V}$ we

have

$$U \leq V \iff V \subseteq U.$$

Remark 1.9. We can identify $\mathcal{F}_x$ as the quotient of the set of pairs (U, s) , where $U \in \mathcal{V}$ and where s is a section of $\mathcal{F}$ on U , by the relation of equivalence defined as follows :

$$(U, s) \sim (V, t) \text{ if and only if there exists an open neighborhood } W \text{ of } x \text{ in } U \cap V \text{ such that } s|_W = t|_W.$$

Moreover, we can see $\mathcal{F}_x$ as the set of sections of $\mathcal{F}$ defined in the neighborhood of x . Two sections belonging to $\mathcal{F}_x$ being considered as equal if they coincide in some neighborhood of x .

Example 1.10. Let $\mathcal{F}(U) = \{ \text{Continuous functions } U \longrightarrow \mathbb{R} \}$. Then $\mathcal{F}_x$ the set of germs of continuous functions at x .

Proposition 1.11. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves, then ψ induces for every point $x \in X$ a morphism $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ between the stalks, where ψ_x is defined by $\psi_x(s_x) = (\psi(U)(s))_x$ for any open subset U of X , $s \in \mathcal{F}(U)$, and $x \in U$.*

Proof. If $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are such that $s_x = t_x$, then there exists an open neighborhood W of x such that $s|_W = t|_W$. So $\psi(U)(s)|_W = \psi(W)(s|_W)$ and $\psi(V)(t)|_W = \psi(W)(t|_W)$. Hence $(\psi(U)(s))_x = (\psi(V)(t))_x$. ■

Note that if $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ and $\phi : \mathcal{G} \longrightarrow \mathcal{H}$ are two morphisms of sheaves we have $(\psi \circ \phi)_x = \psi_x \circ \phi_x$ and $(\text{id}_{\mathcal{F}})_x = \text{id}_{\mathcal{F}_x}$. Moreover, $\psi \longrightarrow \psi_x$ define a functor from the category of sheaves over X to the category $\mathcal{C}$.

Definition 1.12. Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves

- i) We say that ψ is injective if for any open subset U of X , $\psi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective.
- ii) We say that ψ is surjective if for all $x \in X$, $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is surjective.

2. Sheaves

Definition 2.1. We say that a presheaf $\mathcal{F}$ is a sheaf if we have the following properties:

- i) (Uniqueness) Let U be an open subset of X , $s \in \mathcal{F}(U)$, $\{U_i\}_{i \in I}$ a covering of U by open subsets U_i . If $s|_{U_i} = 0$ for every $i \in I$, then $s = 0$.
- ii) (Gluing axiom) If $U = \bigcup_{i \in I} U_i$, and if $s_i \in \mathcal{F}(U_i)$ is a collection of sections matching on the overlaps; that is, they satisfy

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ so that $s|_{U_i} = s_i$, for all $i \in I$

Remarks 2.2. 1) When $\mathcal{F}$ is a presheaf of groups or of an algebraic structure that is in particular a group, we can replace **i**) by: for all $s, t \in \mathcal{F}(U)$ such that for $i \in I$, $s|_{U_i} = t|_{U_i}$ then $s = t$.

2) The section s in **ii**) is unique by condition **i**).

Examples 2.3. 1) Let X be a topological space, $U \mapsto \mathcal{C}^0(U, \mathbb{R})$ is a sheaf of $\mathbb{R}$ -algebras over X .

2) In example 1.5, if moreover, $\mathcal{F}$ is a sheaf then $\mathcal{F}|_U$ is still a sheaf.

2.1. Morphisms of sheaves.

Definition 2.4. A morphism of sheaves is just a morphism of the underlying presheaves.

Remarks 2.5. 1) The sheaves of X form a full subcategory $\mathcal{Sh}_X$ of category of the presheaves on X .

2) The notions injective, surjective and isomorphism for sheaves are defined in the same way as for presheaves.

Lemma 2.6. Let X be a topological space and let U be an open subset of X .

- 1) Let $\mathcal{F}$ be a sheaf on X and let $s, t \in \mathcal{F}(U)$ be two sections such that $s_x = t_x$ for every $x \in U$. Then $s = t$.
- 2) Let $\mathcal{F}, \mathcal{G}$ be presheaves on X and let $\psi, \phi : \mathcal{F} \rightarrow \mathcal{G}$ be morphisms of presheaves on X such that $\mathcal{F}_x = \mathcal{G}_x$ for every $x \in X$. If $\mathcal{G}$ is a sheaf, then $\mathcal{F} = \mathcal{G}$.

Proof. 1) Let $x \in U$, since $s_x = t_x$, there exists an open subset W_x of U containing x such that $s|_{W_x} = t|_{W_x}$. Since $(W_x)_x$ is an open covering of U , according to condition **i**) in definition 2.1, it comes that $s = t$.

2) Let W be an open subset of X and let $s \in \mathcal{F}(W)$. We need to prove that s has the same image under the maps $\psi(W)$ and $\phi(W)$, let $t = \psi(U)(s)$ and $l = \phi(U)(s)$. For all $x \in W$, we have $t_x = \psi_x(s_x) = \phi_x(s_x) = l_x$. Since $\mathcal{G}$ is a sheaf, so by **1**) we get that $t = l$.

■

Proposition 2.7. Let $\psi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then ψ is injective if and only if $\psi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective for every $x \in X$.

Proof. Suppose ψ is injective. Let $x \in X$ and $s_x \in \mathcal{F}_x$ such that $\psi_x(s_x) = 0$, where $s \in \mathcal{F}(U)$ and U is an open neighborhood of x , so $(\psi(U)(s))_x = 0$. Then, there exists an open

neighborhood W of x such that $\psi(U)(s)|_W = 0$ or that $\psi(W)(s|_W) = 0$. From the injectivity of ψ it comes that $s|_W$, thus $s_x = 0$. Conversely, suppose that for all $x \in X$, ψ_x is injective, we fix an open subset V of X and $s \in \mathcal{F}(V)$ such that $\psi(V)(s) = 0$, locally we have, for all $x \in V$, $\psi_x(s_x) = (\psi(U)(s))_x = 0$, it comes from local injectivity, that for all $x \in V$, $s_x = 0$. Hence $s = 0$. ■

Remark 2.8. Proposition 2.7 gives a local characterization of the injectivity.

Theorem 2.9. Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. The following assertions are equivalent :

- 1) ψ is an isomorphism.
- 2) For every $x \in X$, $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is an isomorphism.
- 3) ψ is both injective and surjective.

Proof. 1) $\Rightarrow$ 2) Let ϕ be the inverse morphism of ψ . Plainly, for every $x \in X$, we have $\phi_x \circ \psi_x = \text{id}_{\mathcal{F}_x}$ and $\psi_x \circ \phi_x = \text{id}_{\mathcal{G}_x}$. So ψ_x is an isomorphism.

2) $\Rightarrow$ 3) Immediate, according to proposition 2.7 and definition 1.12, 2)

3) $\Rightarrow$ 1) We will construct the inverse ϕ of ψ . Let W be an open subset of X and $t \in \mathcal{G}(W)$, for every $x \in W$, there exists U_x an open neighborhood of x and $s^x \in \mathcal{F}(U_x)$ such that $t_x = \psi_x(s^x_x) = (\psi(U_x)(s^x))_x$. Hence there exists $V_x \subseteq U_x \cap W$ neighborhood of x such that $t|_{V_x} = (\psi(V_x)(s^x|_{V_x}))|_{V_x}$. If $y \in W$, then $\psi(V_x \cap V_y)(s^x|_{V_x \cap V_y}) = \psi(V_x \cap V_y)(s^y|_{V_x \cap V_y})$, so $s^x|_{V_x \cap V_y} = s^y|_{V_x \cap V_y}$, as the family $(V_x)_{x \in U}$ forms a covering of U , then $(s^x)_x$ rises to a section s of $\mathcal{F}$ on U , and we have $\psi(U)(s) = t$, the uniqueness of s follows from the injectivity of ψ . We set $\phi(U)(t) = s$, then ϕ is the inverse of ψ . ■

3. Sheafification

Definition 3.1. Let $\mathcal{F}$ be a presheaf on a topological space X . We call associated sheaf with $\mathcal{F}$ any sheaf $\mathcal{F}^\dagger$ equipped with a morphism of presheaves $\beta : \mathcal{F} \longrightarrow \mathcal{F}^\dagger$ satisfying the following universal property:

For any morphism of presheaves $\psi : \mathcal{F} \longrightarrow \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there exists a unique morphism of sheaves $\bar{\psi} : \mathcal{F}^\dagger \longrightarrow \mathcal{G}$ such that the following diagram is commutative :

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\psi} & \mathcal{G} \\ \beta \downarrow & \nearrow \bar{\psi} & \\ \mathcal{F}^\dagger & & \end{array}$$

Remark 3.2. The uniqueness of $\mathcal{F}^\dagger$ when it exists is an immediate consequence of the universal property.

Proposition 3.3. *Let $\mathcal{F}$ be a presheaf on a topological space X . Then the sheaf $\mathcal{F}^\dagger$ associated with $\mathcal{F}$ exists and is a unique up to isomorphism. Moreover, using the above notation, all $x \in X$, the induced morphism $\beta : \mathcal{F}_x \longrightarrow \mathcal{F}_x^\dagger$ is an isomorphism.*

Proof. Let $\mathcal{F}$ be a presheaf on X . Consider $Z := \coprod_{x \in X} \mathcal{F}_x$ (disjoint union) and consider the map $\pi : Z \longrightarrow X$ defined by : for all s_x , $\pi(s_x) = x$. For any open V of X and $s \in \mathcal{F}(V)$, let π_s be the map $\pi_s : V \longrightarrow Z$ defined by $\pi_s(x) = s_x$. Note that $\pi(\pi_s(x)) = x$ i.e $\pi \circ \pi_s = \text{id}_V$ (π_s is a section and π is a retraction). We now endow Z with the topology which makes all maps $\pi_s : V \longrightarrow Z$, V open subset of X and $s \in \mathcal{F}(V)$, continuous.

For any open subset V of X , we define $\mathcal{F}^\dagger(V) := \{g : V \longrightarrow Z/g \text{ continuous and } \pi \circ g = \text{id}_V\}$ it is the set of sections of Z on V .

- * For every $W \subseteq V$, the restriction $\mathcal{F}^\dagger(V) \longrightarrow \mathcal{F}^\dagger(W)$ is the usual restriction, i.e $g \longrightarrow g|_W$. In particular $\mathcal{F}^\dagger$ is a presheaf.
- * Condition **i)** in definition 2.1 is immediate.
- * If $(W_j)_j$ is a covering of V and $g_j \in \mathcal{F}^\dagger(W_j)$ are such that for all i, j , $g_i|_{W_i \cap W_j} = g_j|_{W_i \cap W_j}$, then as the g_j are continuous, and coincide on the intersections, there exists $g : V \longrightarrow Z$ which is continuous such that for all j , $g|_{W_j} = g_j$. Moreover g is a section in fact : for all $x \in V$, there is some j such that $x \in W_j$, $\pi \circ g(x) = \pi(g(x)) = \pi(g_j(x)) = x$. $\mathcal{F}^\dagger$ is a sheaf.
- * Definition of $\beta : \mathcal{F} \longrightarrow \mathcal{F}^\dagger$: For any open subset V of X and $s \in \mathcal{F}(V)$, we define $\beta(V)(s) := \pi_s \in \mathcal{F}^\dagger(V)$.
- * Compatibility with restrictions : let $W \subseteq V$ two open subsets of X , $s \in \mathcal{F}(V)$ and $x \in W$, we have $\beta(V)(s)|_W(x) = \pi_s(x) = s_x = (s|_W)(x) = \pi_{s|_W}(x)$. So $\beta(V)(s)|_W = \beta(W)(s|_W)$.
- * Let $\mathcal{G}$ be a sheaf, and $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves. We cut a section g of $\mathcal{F}^\dagger(V)$ into small sections (sections of $\mathcal{F}$) on a covering W_j of V , then by sending them to the $\mathcal{G}(W_j)$, then we stick back into $\mathcal{G}$. Sections of $\mathcal{F}^\dagger$ are obtained by gluing sections of $\mathcal{F}$, so $\mathcal{F}_x = \mathcal{F}_x^\dagger$.

■

Remark 3.4. If $\mathcal{F}$ is a sheaf, it follows from the universal property that $\mathcal{F} \simeq \mathcal{F}^\dagger$.

Example 3.5. Let A be a group (or a ring, an algebra,...), then

$$U \mapsto \begin{cases} A & \text{if } U \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases}$$

is a presheaf and the associated sheaf is called the constant sheaf associated to A . We denoted by $\underline{A}$. For any $x \in X$, we have $\underline{A}_x = A$.

4. Subsheaves and Quotient sheaves

Throughout, we fix a category of objects that have an algebraic structure which are in particular groups, say e.g., $\mathcal{C} = \mathcal{G}p$ or $R\text{-Mod}$.

4.1. Subsheaves.

Definition 4.1. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X , we say that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, if for any open subset U of X , $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and such that we have compatibility with the restrictions induced from $\mathcal{F}$ and $\mathcal{G}$, i.e., For every open subsets $U \subseteq V$ of X , the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}(V) & \hookrightarrow & \mathcal{G}(V) \\ \text{res}_{V,U} \downarrow & & \downarrow \text{res}_{V,U} \\ \mathcal{F}(U) & \hookrightarrow & \mathcal{G}(U) \end{array}$$

Remark 4.2. $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ if, the canonical injection $\iota : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves.

Definition 4.3. Let $\psi : \mathcal{F} \rightarrow \mathcal{G}$ a morphism of presheaves on X . We define the presheaf $\ker(\psi)$ by the formula :

$$U \mapsto \ker(\psi(U))$$

for any open subset U of X . $\ker(\psi)$ is said to be the kernel of ψ , it's a subpresheaf of $\mathcal{F}$. and ψ is injective if and only if its kernel is the trivial presheaf.

Lemma 4.4. Let $\psi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then the presheaf $\ker(\psi)$ is a sheaf.

Proof. Let U be an open of X , $(U_j)_j$ be a covering of U and $s_j \in \ker(\psi(U_j))$ such that for i, j , $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Since $s_j \in \mathcal{F}(U_j)$, then $(s_j)_j$ rises to a section s of $\mathcal{F}$ over U , but for every $x \in U$, there exists i such that $x \in U_j$, and we have $(\psi(U))(s)_x = (\psi(U_j))(s_j)_x = 0$. So $\psi(U)(s) = 0$. Hence $s \in \ker(\psi(U))$. On the other hand, if $s \in \ker(\psi(U))$ such that for every j , $s|_{U_j} = 0$, then $s = 0$ (because $s \in \mathcal{F}(U)$ and $\mathcal{F}$ is a sheaf). ■

Definition 4.5. Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves on X . We define the $\text{im}(\psi)$ presheaf by the formula :

$$U \longmapsto \text{im}(\psi(U))$$

for any open set U of X . One can easily see that $\text{im}(\psi)$ is indeed a subpresheaf of $\mathcal{G}$. We say that $\text{im}(\psi)$ is the image presheaf of ψ .

Remark 4.6. Note that the presheaf $\text{im}(\psi)$ is not in general a sheaf. In the same way we define the presheaf $U \longmapsto \text{coker-pr}(\text{im}(\psi))$ which too is not in general a sheaf. This justifies the following definition.

Definition 4.7. Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaf. The sheaf associated with the image presheaf $\text{im-pr}(\psi)$ called the image sheaf of ψ is denoted $\text{im}(\psi)$. In the same way we define the cokernel sheaf and that we denote by $\text{coker}(\psi)$

Note that in general $(\text{im}(\psi))(U) \neq \text{im}(\psi(U))$. The first term is section of the sheaf $\text{im}(\psi)$ on the open set U , while the second is the image of the morphism $\psi(U)$. More precisely, we have :

Theorem 4.8. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then, the following assertions hold :*

- 1) *For any open subset U of X , and $s \in \mathcal{G}(U)$. $s \in (\text{im}(\psi))(U)$ if and only if there exists an open covering (U_j) of U and $t_j \in \mathcal{F}(U_j)$ such that, for any j , $s|_{U_j} = \psi(U_j)(t_j)$.*
- 2) *ψ is surjective if and only if, for any open subset U of X and $s \in \mathcal{G}(U)$, there exists an open covering $(U_j)_j$ of U and $t_j \in \mathcal{F}(U_j)$ such that, for any j , $s|_{U_j} = \psi(U_j)(t_j)$.*
- 3) *ψ is surjective if and only if $\mathcal{G} = \text{im}(\psi)$.*

Proof. 1) $\text{im}(\psi)$ is a the sheaf associated with presheaf $U \longmapsto \text{im}(\psi(U))$, hence the result.

- 2) If ψ is surjective, let U an open subset of X and $s \in \mathcal{G}(U)$, for all $x \in U$, by theorem 2.9, the map ψ_x is surjective. So there exists $t_x \in \mathcal{F}_x$ such that $\psi_x(t_x) = s_x$. Therefore, there there exists an open neighborhood $U_x \subseteq U$, and $t^x \in U_x$ such that $s|_{U_x} = \psi(U_x)(t^x)$. The covering $(U_x)_{x \in U}$ answers the question. Conversely, let $x \in X$ and $s \in \mathcal{G}(U)$. Let $(U_j)_j$ be covering of U and $t_j \in \mathcal{F}(U_j)$ such that $s|_{U_j} = \psi(U_j)(t_j)$ for all j . Since $\mathcal{F}$ is a sheaf then there is $t \in \mathcal{F}(U)$ such that $t|_{U_j} = t_j$ for all j . In particular, for every j such that $x \in U_j$, $s_x = (s|_{U_j})_x = (\psi(U_j)(t_j))_x = \psi_x(t_j)$. Hence ψ is surjective.

- 3) Immediate from 1) and 2).

■

4.2. Quotients sheaves. Assume that $\mathcal{F}$ is a subsheaf of the sheaf $\mathcal{G}$. Then we can define a presheaf whose sections over U are the quotient $\mathcal{G}(U)/\mathcal{F}(U)$. The restriction maps of $\mathcal{F}$ and $\mathcal{G}$

are compatible the inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and hence pass to the quotient $\mathcal{G}(U)/\mathcal{F}(U)$. This presheaf, i.e., $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$, is called quotient presheaf of $\mathcal{G}$ by $\mathcal{F}$.

Definition 4.9. The quotient sheaf $\mathcal{G}/\mathcal{F}$ is the sheafification of the quotient presheaf of $\mathcal{G}$ by $\mathcal{F}$.

Proposition 4.10. Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$, $x \in X$. Then $(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x$.

Proof. $\mathcal{G}/\mathcal{F}$ is the sheaf associated with the presheaf $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$ whose stalks at x is clearly isomorphic to $\mathcal{G}_x/\mathcal{F}_x$. ■

5. Continuous maps and sheaves

5.1. Pushforward.

Definition 5.1. Let $f : Y \rightarrow X$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a presheaf on X . We define the pushforward of $\mathcal{F}$ by the formula :

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for any open $V \subseteq Y$.

Given opens $W \subseteq V$ of Y open the restriction map is given by the commutativity of the diagram

$$\begin{array}{ccc} f_*\mathcal{F}(V) & \xlongequal{\quad} & \mathcal{F}(f^{-1}(V)) \\ \downarrow & & \downarrow \text{res}_{f^{-1}(V), f^{-1}(W)} \\ f_*\mathcal{F}(W) & \xlongequal{\quad} & \mathcal{F}(f^{-1}(W)) \end{array}$$

It is clear that this defines a presheaf on Y .

Remark 5.2. The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$\begin{array}{ccc} f_* : \mathcal{P}\text{reSh}_X & \longrightarrow & \mathcal{P}\text{reSh}_Y \\ \mathcal{F} & \longmapsto & f_*\mathcal{F} \end{array}$$

Proposition 5.3. Let $f : X \rightarrow Y$ be a continuous map and $\mathcal{F}$ be a sheaf on X . Then $f_*\mathcal{F}$ is a sheaf on Y .

Proof. This immediately follows from the fact that if $(W_j)_j$ is an open covering of some open subset W of Y then, $\bigcup_j f^{-1}(W_j)$ is an open covering of the open $f^{-1}(W)$. Consequently, we obtain a functor

$$f_* : \mathcal{S}h_X \longrightarrow \mathcal{S}h_Y$$

This is compatible with composition in the following strong sense : ■

Lemma 5.4. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps of topological spaces. Then, the functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal.*

Proof. Immediate. ■

5.2. Pullback. We saw in example 1.5 that if $\mathcal{F}$ is a sheaf on X , then for any open subset U of X $\mathcal{F}|_U$ is a sheaf on U . Now if we take an arbitrary subset Z of X . the restriction of $\mathcal{F}$ on Z is not necessarily a sheaf because an open set W of Z is not necessarily an open set of X . Next definition gives the meaning of $\mathcal{F}|_Z$, when Z is a closed subset of X . This will be generalized in Definition 5.6 to give the meaning of the pullback presheaf defined by a continuous map. For this purpose, note that if $f : X \longrightarrow Y$ is a continuous map between topological spaces and V is an open of Y , then the family $(U)_{f(U) \subseteq V}$ consisting of all open subsets U of X satisfying $f(U) \subseteq V$, is an inductive system for the inverse of the inclusion relation.

Definition 5.5. If $\iota : Z \longrightarrow X$ is the inclusion of a closed subset Z of X , and V is an open subset of Z . We define the restriction $\mathcal{F}|_Z$ as the sheafification of the following presheaf

$$V \longmapsto \varinjlim_{V \subseteq U} \mathcal{F}(U)$$

Definition 5.6. Let $f : X \longrightarrow Y$ be a continuous map between topological spaces and $\mathcal{G}$ be a presheaf on Y . We define the pullback presheaf of $\mathcal{G}$ by the formula :

$$f_p \mathcal{G}(U) = \varinjlim_{f(U) \subseteq V} \mathcal{G}(V).$$

Remark 5.7. In the language of categories. The pullback presheaf $f_p \mathcal{G}$ of $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words, $f_p \mathcal{G}$ will be a presheaf on X such that

$$\text{Mor}_{\mathcal{P}\text{reSh}_X}(f_p \mathcal{G}, \mathcal{F}) = \text{Mor}_{\mathcal{P}\text{reSh}_Y}(\mathcal{G}, f_* \mathcal{F})$$

Proposition 5.8. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces, x be a point of X and $\mathcal{G}$ be a presheaf on Y . Then, up to an isomorphism, we have $(f_p \mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof.

$$\begin{aligned} (f_p \mathcal{G})_x &= \varinjlim_{x \in U} f_p \mathcal{G}(U) \\ &= \varinjlim_{x \in U} \varinjlim_{f(U) \subseteq V} \mathcal{G}(V) \\ &= \varinjlim_{f(x) \in V} \mathcal{G}(V) \\ &= \mathcal{G}_{f(x)} \end{aligned}$$

■

Definition 5.9. Let $f : X \longrightarrow Y$ be a continuous map between topological spaces and $\mathcal{G}$ be a sheaf on Y . The pullback sheaf $f^{-1}\mathcal{G}$ is defined by the formula :

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^\dagger$$

$f^{-1}\mathcal{G}$ is also called the inverse image along the map f .

Remark 5.10. f^{-1} defines a functor :

$$\begin{array}{ccc} f^{-1} : \mathcal{Sh}_Y & \longrightarrow & \mathcal{Sh}_X \\ \mathcal{G} & \longmapsto & f^{-1}\mathcal{G} \end{array}$$

The pullback f^{-1} is a left adjoint of pushforward on sheaves.

$$\text{Mor}_{\mathcal{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{\mathcal{Sh}_Y}(\mathcal{G}, f_*\mathcal{F})$$

Example 5.11. Let $\mathcal{F}$ be a sheaf on X and $x \in X$. Let $\iota : \{x\} \longrightarrow X$ be the inclusion map, then $\iota^{-1}\mathcal{F} = \mathcal{F}_x$

Lemma 5.12. Let $f : X \rightarrow Y$ be a continuous map between topological spaces, $x \in X$ and $\mathcal{G}$ be a sheaf on Y , then the stalks $(f^{-1}\mathcal{G})_x$ and $\mathcal{G}_{f(x)}$ are equals.

Proof. This a combination of proposition 3.3 and proposition 5.8. ■

Lemma 5.13. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly, $(g \circ f)_p = f_p \circ g_p$, for presheaves.

Proof. This follows from the fact that adjoint functors are unique up to unique isomorphism, and Lemma 5.4. ■

6. Exact sequences of sheaves

Definition 6.1. A sequence of presheaves with presheaves morphisms

$$\dots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^j \xrightarrow{\psi^j} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \dots$$

is said to be exact if for all i , $\text{im}(\psi^{j-1}) = \ker(\psi^j)$. In particular the following exact sequence is call a short exact sequence when it is exact :

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

Remark 6.2. Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then by,

i) ψ is injective if and only if

$$0 \longrightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{G}$$

is an exact sequence.

ii) ψ is surjective if and only if

$$\mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow 0$$

is an exact sequence.

Example 6.3. Let $X = \mathbb{C}$, and $\mathcal{O}_X$ the sheaf of holomorphic functions and consider the map $d : \mathcal{O}_X \rightarrow \mathcal{O}_X$, sending $f(z)$ to $f'(z)$. There is an exact sequence

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \xrightarrow{d} \mathcal{O}_X \longrightarrow 0$$

Indeed,

- * A function whose derivative vanishes identically is locally constant, so $\ker(d)$ is the constant sheaf $\mathbb{C}_X$.
- * In small open disks any holomorphic function is a derivative.

Lemma 6.4. Let $\psi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on X . Then for any $x \in X$, we have $(\ker \psi)_x = \ker(\psi_x)$ and $(\operatorname{im} \psi)_x = \operatorname{im}(\psi_x)$.

Proof. Let $s_x \in (\ker(\psi))_x$, and let U an open neighborhood of x such that $s \in (\ker(\psi))(U) = \ker(\psi(U))$, so $\psi(U)(s) = 0$, hence $\psi_x(s_x) = (\psi(U)(s))_x = 0$, so $s_x \in \ker(\psi_x)$. Conversely, if $\psi_x(s_x) = 0$, then $(\psi(U)(s))_x = 0$ (U is an open neighborhood of x and $s \in \mathcal{F}(U)$), then there exists an open neighborhood $W \subseteq U$ of x such that $\psi(U)(s)|_W = 0$, it comes while $\psi(W)(s|_W) = 0$ and therefore $s|_W \in \ker(\psi(W))$ whence $s_x = (s|_W)_x \in (\ker(\psi))_x$. One can proceed similarly for the image. ■

Theorem 6.5. A sequence of sheaves with sheaves morphisms

$$\dots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^j \xrightarrow{\psi^j} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \dots$$

is an exact sequence if and only if for any $x \in X$

$$\dots \longrightarrow \mathcal{F}_x^{j-1} \xrightarrow{\psi_x^{j-1}} \mathcal{F}_x^j \xrightarrow{\psi_x^j} \mathcal{F}_x^{j+1} \xrightarrow{\psi_x^{j+1}} \dots$$

is an exact sequence.

Proof. Immediate from the preceding lemma. ■

Proposition 6.6. *Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$ on X . Then*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0$$

is exact sequence.

Proof. By proposition 4.10, for any $x \in X$,

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{G}_x/\mathcal{F}_x = (\mathcal{G}/\mathcal{F})_x \longrightarrow 0$$

is exact sequence. Hence the result. ■

Remark 6.7. If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence over X , then $\mathcal{F}$ identified with a sub-sheaf of $\mathcal{G}$ and $\mathcal{G}/\mathcal{F} \simeq \mathcal{H}$.

Corollary 6.8. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then*

- 1) $\text{im}(\psi) \simeq \mathcal{F}/\ker(\psi)$.
- 2) $\text{coker}(\psi) \simeq \mathcal{G}/\text{im}(\psi)$.

Proof. 1) It is easy to check that for all $x \in X$, we have

$$0 \longrightarrow (\ker(\psi))_x \longrightarrow \mathcal{F}_x \longrightarrow \text{im}(\psi)_x \longrightarrow 0$$

It follows by theorem 6.5, that

$$0 \longrightarrow \ker(\psi) \longrightarrow \mathcal{F} \longrightarrow \text{im}(\psi) \longrightarrow 0$$

is an exact sequence. Also by remark 6.7 we have $\text{im}(\psi) \simeq \mathcal{F}/\ker(\psi)$

- 2) Similar to 1).

■

7. Glueing sheaves

In this section, we fix a topological space X , and we consider an open covering $\{U_i\}_{i \in I}$ of X with a sheaf $\mathcal{F}_i$ on each subset U_i . Our goal is to “glue” the $\mathcal{F}_i$ together, that is we search for a global sheaf $\mathcal{F}$ such that $\mathcal{F}|_{U_i} = \mathcal{F}_i$ for all $i \in I$.

Notation. i) For $i, j \in I$, we denote by U_{ij} the intersection $U_i \cap U_j$.

ii) For $i, j, k \in I$, we denote by U_{ijk} the intersection $U_i \cap U_j \cap U_k$.

Definition 7.1. A Gluing Datum consists of a family of sheaves $\mathcal{F}_i$ over U_i and a family of morphisms $\delta_{ij} : \mathcal{F}_i|_{U_{ij}} \longrightarrow \mathcal{F}_j|_{U_{ij}}$ such that

- i) $\delta_{ii} = \text{id}_{\mathcal{F}_i}$.

$$\text{ii) } \delta_{ji} = \delta_{ij}^{-1}.$$

$$\text{iii) } \delta_{ik} = \delta_{jk} \circ \delta_{ij} \text{ on } U_{ijk}.$$

A morphism of gluing datum $(\mathcal{F}_i, \delta_{ij}) \longrightarrow (\mathcal{G}_i, \eta_{ij})$ is a family of morphism of sheaves $\psi_i : \mathcal{F}_i \longrightarrow \mathcal{G}_i$ such that the following diagram

$$\begin{array}{ccc} \mathcal{F}_i & \xrightarrow{\psi_i} & \mathcal{G}_i \\ \delta_{ij} \downarrow & & \downarrow \delta_{ij} \\ \mathcal{F}_j & \xrightarrow{\psi_j} & \mathcal{G}_j \end{array}$$

is commutative.

Theorem 7.2 (*Gluing sheaves*). *There exists a sheaf $\mathcal{F}$ on X , unique up to isomorphism such that there are isomorphisms $\theta_i : \mathcal{F}_{U_i} \longrightarrow \mathcal{F}_i$ such that there are satisfying*

$$\theta_j = \delta_{ij} \circ \theta_i.$$

Proof. Let W be an open subset of X . We write $W_i = U_i \cap W$, and $W_{ij} = U_{ij} \cap W$. We are going to define the sections of $\mathcal{F}$ over W by gluing sections of the $\mathcal{F}'_i$ s over W'_i s along the W'_{ij} s using the isomorphisms δ_{ij} . We define

$$\mathcal{F}(W) := \{(s_i)_{i \in I} \mid \delta_{ji}(s_i|_{W_{ij}}) = s_j|_{W_{ij}}\} \subseteq \prod_{i \in I} \mathcal{F}_i(W_i). \quad (7.1)$$

The δ_{ij} 's are morphisms of sheaves and therefore are compatible with all restrictions maps (see definition 1.6). So if $V \subseteq W$ is another open subset we have

$$\delta_{ij}(s_i|_{W_{ij}}) = s_j|_{W_{ij}}.$$

Because of this, the defining condition (7.1) is compatible with componentwise restrictions, and they can therefore be used as the restriction maps in $\mathcal{F}$. So We have defined a presheaf on X .

- * The first step: is to establish the isomorphisms $\theta_i : \mathcal{F}_{U_i} \longrightarrow \mathcal{F}_i$. To avoid getting confused by the names of the indices, we shall work with a fixed index $j \in I$. Suppose $W \subseteq U_j$ is an open set. We have $W = W_j$, and projecting from the product $\prod_{i \in I} \mathcal{F}_i(W_i)$ onto the component

$$\mathcal{F}_j(W) = \mathcal{F}_j(W_j)$$

gives us a map $\theta : \mathcal{F}_j|_{W_j} \longrightarrow \mathcal{F}_j$. Moreover, $\theta((s_i)_{i \in I}) = s_j$. The situation is summarized in the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}(W) & \hookrightarrow & \prod_{i \in I} \mathcal{F}_i(W_i) \\ & \searrow \theta_j & \downarrow \pi_j \\ & & \mathcal{F}_j(W) \end{array}$$

Now, we want to show that θ_j 's give the desired isomorphisms. We note that on the restrictions $W_{jj'}$, the requirement in the proposition, that

$$\theta_{j'} = \eta_{j'j} \circ \theta_j$$

is fulfilled. This follows directly from the (7.1) that

$$s_j|_{W_{jj'}} = \delta_{jj'}(s_{j'}|_{W_{jj'}}).$$

- * θ_j is surjective: Let α a section of $\mathcal{F}_j(W)$ over some $W \subseteq U_j$, and pose $s = (\delta_{ij}(\alpha|_{W_{ij}}))_{i \in I}$. Then s satisfies (7.1) and is an element $\mathcal{F}(W)$. Indeed, by definition 7.1 iii) we obtain

$$\delta_{ki}(\delta_{ij}(\alpha|_{W_{kij}})) = \delta_{kj}(\alpha|_{W_{kij}}).$$

for each $i, k \in I$, and that is just the condition (7.1). As $\delta_{jj}(\alpha|_{W_{jj}}) = \alpha$ by the first gluing request, the element s projects to the section α of $\mathcal{F}_j$.

- * θ_j is injective : Since $s_j = 0$ it follows that $s_i|_{W_{ij}} = \delta_{ij}(s_j) = 0$ for each $i \in I$. Now $\mathcal{F}_j$ is a sheaf, and the $\{V_{ij}\}_{i \in I}$ constitute an open covering of W_j , so we may conclude that $s = 0$ by definition 2.1 i).
- * The final step: To show that $\mathcal{F}$ is a sheaf. Let $\{W_j\}_{j \in I}$ be an open covering of $W \subseteq U$, and $s_j \in \mathcal{F}(W_j)$ is a bunch of sections matching on the intersections $W_{jj'}$. Since $\mathcal{F}|_{U_i \cap W}$ is a sheaf patch together to give sections s_i in $\mathcal{F}_{U_i \cap W}$ matching on the overlaps $U_{ij} \cap W$. This last condition means that $\delta_{ij}(s_i) = s_j$. By definition $(s_i)_{i \in I}$, then is a section in $\mathcal{F}(W)$ restricting to s_i . Hence the result.

The Gluing axiom (see definition 2.1) is easier : Let $s = (s_i)_{i \in I}$ in $\mathcal{F}(W)$, and a covering $\mathcal{L} = \{V_j\}_{j \in J}$ of W such that $s|_{V_j} = 0$ for all $j \in J$, then also $s|_{V_j \cap W_i} = 0$, and since $\{V_j \cap W_i\}_{j \in J}$ forms a covering of W_i , we must have $s|_{W_i} = 0$ as well, since $\mathcal{F}_{W_i} = \mathcal{F}_i$ is a sheaf. But from the (7.1) we thus see that $s = 0$.

■

YASSINE AIT MOHAMED

Sidi Mohamed Ben Abdellah, Fez, Morocco

e-mail: y.aitmohamed@yahoo.com